Final exam review

New stuff: Fourier series
Laplace transform
Fourier transform

Using these methods to solve differential equations:

1. transform original diff. eqn
2. solve it
3. impose initial conditions
4. transform back

example 1: QM particle in a box of size L

Let the initial wavefunction be $\Psi(x,0) = N x(L-x)$
What is $\Psi(x,t)$?

Boundary conditions $\Psi(0,t) = \Psi(L,t) = 0 \rightarrow$ expand Ψ as

$$\Psi(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{\pi n x}{L}\right)$$

Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) = +i\hbar\frac{\partial \Psi(x,t)}{\partial t}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty} a_n(t)\sin\left(\frac{\pi n x}{L}\right) = i\hbar\sum_{n=1}^{\infty}\frac{\partial a_n}{\partial t}\sin\left(\frac{\pi n x}{L}\right)$$

Act on both sides with $\frac{2}{L}\int_0^L dx \sin\left(\frac{\pi k x}{L}\right)$ and use $\frac{2}{L}\int_0^L dx \sin\left(\frac{\pi n x}{L}\right)\sin\left(\frac{\pi k x}{L}\right) = \delta_{nk}$

$$+\frac{\hbar^2}{2m} a_k(t)\left(\frac{\pi k}{L}\right)^2 = i\hbar\frac{\partial a_k}{\partial t} \qquad \text{(step 1)}$$

$$\frac{\partial a_n}{\partial t} = -i\underbrace{\frac{\hbar\pi^2 k^2}{2mL^2}}_{\omega_k = \frac{\hbar\pi^2k^2}{2mL^2}} a_k(t) = -i\omega_k a_k(t)$$

So $a_n(t) = a_n(0)\, e^{-i\omega_n t}$ (step 2)

Initial condition

$$\Psi(x,0) = \sum_{n=1}^{\infty} a_n(0)\sin\left(\frac{\pi n x}{L}\right) = N x(L-x)$$

Act on both sides with $\frac{2}{L}\int_0^L dx \sin\left(\frac{k\pi x}{L}\right)$:

$$a_k(0) = \frac{2}{L}\int_0^L dx \sin\left(\frac{k\pi x}{L}\right) N x(L-x)$$

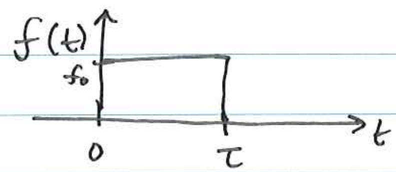
$$= \begin{cases} 0 & k = \text{even} \\ N\frac{8L^2}{k^3\pi^3} & k = \text{odd} \end{cases} \qquad \text{(step 3)}$$

$$\Psi(x,t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{8NL^2}{n^3\pi^3}\sin\left(\frac{n\pi x}{L}\right) \times e^{-i\omega_n t} \qquad \text{(step 4)}$$

$$\Psi(x,t) = \frac{8NL^2}{\pi^3} \sin\left(\frac{x\pi}{L}\right) e^{-i\omega_1 t} + \frac{8NL^2}{9\pi^3} \sin\left(\frac{3\pi x}{L}\right) e^{-i\omega_3 t} + \dots$$

example 2: simple harmonic oscillator with forcing term

$$\ddot{x}(t) + \omega^2 x(t) = f(t) = \begin{cases} f_0 & 0 < t < \tau \\ 0 & \text{otherwise} \end{cases}$$



initial conditions: $x(0) = \dot{x}(0) = 0$.

Laplace transform differential equation:

$$X(s) = \int_0^\infty dt\, e^{-st} x(t)$$

$$\left(s^2 X(s) - s\,x(0) - \dot{x}(0)\right) + \omega^2 X(s) = F(s) \qquad \text{(step 1)}$$

where $F(s) = \int_0^\infty dt\, e^{-st} f(t) = \int_0^\tau dt\, f_0\, e^{-st}$

$$= f_0 \left(\frac{e^{-st}}{-s}\right)\Bigg|_0^\tau = \frac{f_0}{s}\left(1 - e^{-s\tau}\right)$$

So we have:

$$X(s) = \frac{1}{s^2+\omega^2}\left(F(s) + s\,x(0) + \dot{x}(0)\right) \qquad \text{(step 2)}$$

Impose initial condition $x(0) = \dot{x}(0) = 0$

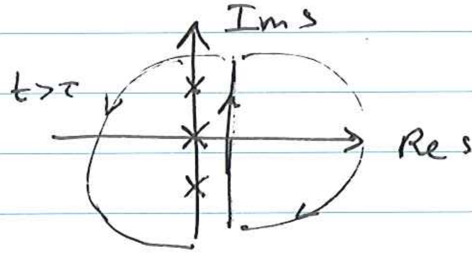
$$X(s) = \frac{F(s)}{s^2+\omega^2} \qquad \text{(step 3)}$$

Transform back:

$$X(s) = \frac{f_0}{s}\frac{1}{(s+i\omega)(s-i\omega)}\left(1-e^{-s\tau}\right)$$

Use Bromwich integral: poles at $s=0, \pm i\omega$

$$x(t) = \frac{1}{2\pi i}\int_{-i\infty+\gamma}^{+i\infty+\gamma} ds\; f_0 \frac{1}{s(s+i\omega)(s-i\omega)}\left(1-e^{-s\tau}\right)e^{st}$$



First consider the 2nd term:

$$\frac{1}{2\pi i}\int_{-i\infty+\gamma}^{+i\infty+\gamma} ds\; f_0 \frac{e^{s(t-\tau)}}{s(s+i\omega)(s-i\omega)} = 0 \quad \text{for } t<\tau.$$

$$= \frac{1}{2\pi i}\times 2\pi i\, f_0 \sum \text{Residues} \qquad \text{for } t>\tau$$

$$= f_0\left(\frac{1}{(i\omega)(-i\omega)} + \frac{e^{i\omega(t-\tau)}}{i\omega(2i\omega)} + \frac{e^{-i\omega(t-\tau)}}{(-i\omega)(-2i\omega)}\right)$$

$$= \frac{f_0}{\omega^2}\left(1 - \frac{1}{2}e^{i\omega(t-\tau)} - \frac{1}{2}e^{-i\omega(t-\tau)}\right)$$

$$= \frac{f_0}{\omega^2}\left(1-\cos\left(\omega(t-\tau)\right)\right)$$

$$= \frac{f_0}{\omega^2}\sin^2\left(\frac{\omega(t-\tau)}{2}\right)$$

1st term is the same with $\tau \to 0$.

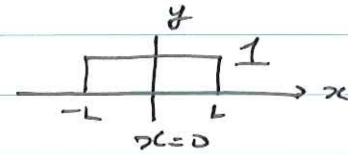
$$x(t) = \frac{f_0}{\omega^2}\left(1-\cos(\omega t) - 1 + \cos(\omega(t-\tau))\right)$$

$$= \frac{f_0}{\omega^2}\left(\cos(\omega(t-\tau)) - \cos(\omega t)\right) \qquad \text{(step 4)}$$

example: infinite string

Wave equation $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$

Suppose we have initial condition



$$y(x,0) = \begin{cases} 0 & |x| > L \\ 1 & |x| < L \end{cases}$$

$$Y(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} y(x,t)$$

$$\ddot{Y}(k,t) = -v^2 k^2 Y(k,t) \qquad \text{let } \omega = vk \qquad \text{(step 1)}$$

$$Y(k,t) = A(k) e^{-i\omega t} + B(k) e^{i\omega t} \qquad \text{(step 2)}$$

$$Y(k,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-ikx} y(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} dx\, e^{-ikx}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{-ik} e^{-ikx} \Big|_{-L}^{L} = \frac{1}{-ik\sqrt{2\pi}} \left(e^{-ikL} - e^{ikL}\right)$$

$$\dot{Y}(k,0) = 0$$

So $$A + B = \frac{1}{-ik\sqrt{2\pi}} \left(e^{-ikL} - e^{+ikL}\right) = 2A = 2B$$

$$-i\omega(A - B) = 0 \rightarrow A = B$$

$$A(k) = B(k) = \frac{1}{+2ik\sqrt{2\pi}} \left(e^{ikL} - e^{-ikL}\right)$$

$$Y(k,t) = \frac{1}{2ik\sqrt{2\pi}} \left(e^{ikL} - e^{-ikL}\right)\left(e^{-i\omega t} - e^{i\omega t}\right)$$

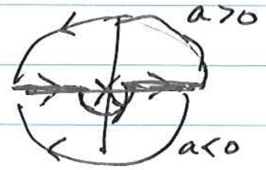
(Step 3)

$$y(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx}\, Y(k,t)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{1}{2ik} \left(e^{ikL} - e^{-ikL}\right)\left(e^{-ivkt} - e^{ivkt}\right) e^{ikx}$$

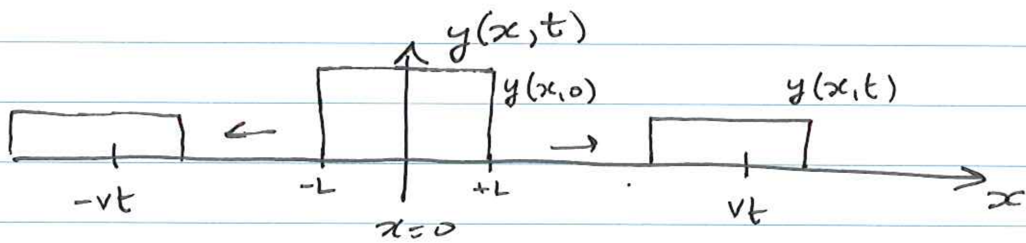
$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \frac{1}{2k} \Big(e^{i(x+L-vt)k} - e^{i(x+L+vt)k} - e^{i(x-L-vt)k} + e^{i(x-L+vt)k} \Big)$$

Note: $\displaystyle \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \frac{1}{2k} e^{iak} = \begin{cases} \frac{1}{2}\left(1 - \frac{1}{2}\right) & a > 0 \\ \frac{1}{2}\left(-\frac{1}{2}\right) & a < 0 \end{cases}$



$$= \frac{1}{2}\theta(a) - \frac{1}{4}$$

$$y(x,t) = \frac{1}{2}\Big(\theta(x+L-vt) - \theta(x+L+vt) - \theta(x-L-vt) + \theta(x-L+vt)\Big)$$



$$\theta(x+L-vt) - \theta(x-L-vt) = \begin{cases} 1 & \text{if } x-L < vt < x+L \\ 0 & \text{otherwise} \end{cases}$$